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A FITTED MESH METHOD FOR A COUPLED SYSTEM OF TWO-PARAMETER FOURTH-ORDER SINGULARLY PERTURBED REACTION–DIFFUSION PROBLEMS: STABILITY AND CONVERGENCE ANALYSIS

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Abstract: We study a coupled system of two fourth-order singularly perturbed reaction–diffusion boundary value problems, in which each equation contains two independent small parameters ε_i and μ_i (i = 1, 2) multiplying the fourth- and second-order derivative terms, respectively, together with a cooperative zero-order coupling between the two components. Such systems arise as simplified models of coupled elastic beams / plates with a small flexural rigidity and a small elastic-foundation response. By introducing the moment variables v_i = −y_i″, the problem is reduced to a weakly coupled system of four second-order equations. An energy (L^2) stability estimate, uniform in ε_i and μ_i, is derived for the continuous problem, and its discrete analogue is obtained via a summation-by-parts identity for the classical three-point non-uniform second-difference operator. A piecewise-uniform (Shishkin-type) fitted mesh, with a transition parameter determined by the effective decay rate κ_i = √(μ_i α_i/ε_i) of each equation, is constructed to resolve the boundary layers at both endpoints. The resulting finite difference scheme is implemented and validated in two stages: first, a manufactured-solution test in the regular (non-stiff) regime confirms second-order accuracy of the implementation; second, a genuine singularly perturbed experiment, using the double-mesh principle over nine decades of ε (10^{-2} to 10^{-10}) and mesh sizes N = 32,…,512, shows that the maximum nodal error on the fitted mesh stabilizes as ε → 0 (the hallmark of parameter-uniform convergence), with an observed order between 0.88 and 1.34, consistent with the expected $O(N^{-1} \ln N)$ theoretical rate. The same double-mesh diagnostic on a uniform mesh is shown to be numerically misleading for very small ε, which is discussed as a cautionary methodological point. To the best of our search of the recent literature, the specific combination treated here — fourth order, two independent small parameters, and a coupled system — has not been previously addressed, and existing work treats these three features only separately.

Keywords: singularly perturbed problems; two-parameter problems; fourth-order boundary value problems; fitted mesh method; Shishkin mesh; parameter-uniform convergence; coupled systems; finite difference method.

1. Introduction

Singularly perturbed differential equations, in which a small parameter multiplies the highest-order derivative, are ubiquitous in the mathematical modelling of diffusion–reaction processes, elasticity, fluid dynamics and semiconductor theory. Their solutions typically exhibit thin boundary or interior layers, and standard numerical methods on uniform meshes fail to converge uniformly with respect to the perturbation parameter; this has motivated an extensive literature on fitted mesh and fitted

operator methods, beginning with the foundational monographs of Miller, O'Riordan and Shishkin [1] and Roos, Stynes and Tobiska [2], and the layer-adapted mesh theory of Linß [3].

Fourth-order singularly perturbed boundary value problems arise naturally in the bending of thin elastic beams and plates resting on an elastic foundation, where the flexural rigidity is small relative to the foundation stiffness. The standard approach reduces the fourth-order equation to a coupled pair of second-order equations via a moment (or curvature) variable, after which fitted-mesh finite difference or spline techniques designed for second-order problems can be adapted. Singh and Kumar [4] developed a spline-based parameter-uniform scheme for a single fourth-order singularly perturbed equation with one small parameter, establishing almost first-order uniform convergence on a Shishkin mesh, and Reda, Bullo and Duressa [5] proposed a related fourth-order fitted-mesh scheme for semilinear singularly perturbed reaction–diffusion problems.

A separate and very active line of research concerns two-parameter singularly perturbed problems, in which two independent small parameters multiply the highest- and an intermediate-order derivative term. Such problems exhibit a richer layer structure than the single-parameter case, classified into three regimes according to the relative size of the two parameters, and require a correspondingly more delicate fitted-mesh construction. Recent contributions include the extended cubic B-spline collocation method of Negero [6] for two-parameter time-delayed convection–diffusion problems; the quadratic B-spline collocation method of Shivhare, Podila and Kumar [7] for two-parameter parabolic problems; the almost-second-order fitted mesh method of Chandru, Prabha, Shanthi and Ramos [8] for two-parameter problems with a discontinuous convection coefficient; the n-equation coupled system treatment of Mathiyazhagan, Arthur, Chatzarakis and Panetsos [9]; and the turning-point-with-delay analysis of Singh, Kumar and Vigo-Aguiar [10]. Further recent two-parameter studies include the non-polynomial spline method of Prathap and Rao [11] for problems with mixed shifts, the nonstandard finite difference scheme of Mohye et al. [12], the third-order delay-equation treatment of Rajendran et al. [13], the exponential-spline method of Zahra and El Mhlawy [14], the streamline-diffusion finite element method of Avijit and Natesan [15], and the weak Galerkin method on Bakhvalov-type meshes of Toprakseven, Kaushik and Sharma [16].

A third relevant strand concerns coupled systems of singularly perturbed equations, in which the interaction between components — rather than a second small parameter — is itself a source of additional layer complexity. Liu, Long and Zhang [17] established parameter-uniform convergence for a coupled system of parabolic convection–diffusion equations, and Jaiswal, Kumar and Clavero [18] proposed an efficient component-wise splitting approach for coupled parabolic reaction–diffusion systems with interior layers. Basha and Shanthi analysed weakly coupled systems of convection–diffusion type with discontinuous source terms, both for Robin-type boundary conditions [19] and for the corresponding reaction–convection–diffusion problem [20], and Mariappan, Chandru and Ramos [21] developed a second-order robust method for a partially singularly perturbed time-dependent reaction–diffusion system.

The broader fitted-mesh and layer-adapted literature that informs the present construction includes higher-order schemes such as the quintic-spline Shishkin-mesh method of Arumugam, Thynesh, Chandru and Ramos [22] for periodic problems; the turning-point analysis of Ranjan and Prasad [23]; the twin-boundary-layer parabolic treatment of Bullo and Kusi [24]; the Bakhvalov-type mesh construction used by Kumar, Kumar, Ramos and Kuldeep [25] for degenerate parabolic convection–diffusion problems; the parameter-robust method for semilinear parabolic convection–diffusion equations of Yadav and Mukherjee [26]; the delay-differential-equation treatment with non-local boundary conditions of Debela and Duressa [27]; the a priori/a posteriori error analysis for parameterized problems of Kumar, Kumar and Das [28]; the higher-order fitted-mesh method for parabolic reaction–diffusion problems of Duressa, Gelu and Kebede [29]; and singularly perturbed Fredholm integro-differential equations, treated by Elango et al. [30], Durmaz, Amirali and Amiraliyev [31], and Govindarao, Al-Ghafri, Mohapatra and Nhan [32]. A layer-adapted method for functional-differential equations [33] and a convergence analysis of the local discontinuous Galerkin

method for reaction–diffusion problems [34] further illustrate the range of numerical-analysis tools developed for this class of problems.

Despite this substantial body of work, a systematic search of the recent literature (2023–2026) did not locate any study that combines all three of the following features simultaneously: (i) a fourth-order differential operator, (ii) two independent small parameters, and (iii) a coupled system of two or more such equations. Each feature individually is well studied; their combination is not, indicating a specific, well-defined gap consistent with the incremental but rigorous style of contribution typical of this literature.

The contribution of this paper is threefold. First, we formulate a coupled system of two fourth-order reaction–diffusion equations with independent two-parameter perturbations and establish, via an energy method, a stability estimate that is uniform in both parameters of both equations. Second, we construct a Shishkin-type fitted mesh, driven by the effective decay rate of each equation, and a corresponding finite-difference scheme, for which we prove a discrete stability estimate using a summation-by-parts identity. Third, we implement the method and verify it numerically in two independent ways: a manufactured-solution correctness test, and a genuine singularly perturbed experiment spanning nine decades of the perturbation parameter, which confirms the expected parameter-uniform convergence behaviour and, as a secondary methodological contribution, exposes and documents a misleading-error artefact of the double-mesh technique when it is (incorrectly) applied on a uniform mesh for very small ε.

The remainder of the paper is organized as follows. Section 2 states the problem and the standing assumptions. Section 3 performs the reduction to a second-order system. Section 4 establishes the continuous energy stability estimate. Section 5 describes the layer structure and the solution decomposition. Sections 6–8 construct the fitted mesh, the discrete scheme, and its discrete stability estimate. Section 9 states the parameter-uniform convergence theorem. Section 10 reports the numerical experiments, and Section 11 concludes.

2. Problem Formulation and Assumptions

For i = 1, 2 and x ∈ Ω = (0,1), consider the coupled system

$$\boldsymbol{\varepsilon_i}\, \boldsymbol{y_i^{(4)}}(\boldsymbol{x}) \;-\; \boldsymbol{\mu_i}\, \boldsymbol{a_i}(\boldsymbol{x})\, \boldsymbol{y_i''}(\boldsymbol{x}) \;+\; \sum_{\boldsymbol{j=1}}^{\boldsymbol{2}} \boldsymbol{b_{ij}}(\boldsymbol{x})\, \boldsymbol{y_j}(\boldsymbol{x}) \;=\; \boldsymbol{f_i}(\boldsymbol{x}),$$

subject to the simply-supported (moment-free) boundary conditions

$$\boldsymbol{y_i}(\boldsymbol{0}) \;=\; \boldsymbol{y_i}(\boldsymbol{1}) \;=\; \boldsymbol{0},\;\; \boldsymbol{y_i''}(\boldsymbol{0}) \;=\; \boldsymbol{y_i''}(\boldsymbol{1}) \;=\; \boldsymbol{0},\;\; \boldsymbol{i} \;=\; \boldsymbol{1,2},$$

where 0 < ε_i ≤ 1 and 0 < μ_i ≤ 1 are two independent small parameters for each equation. We make the following standing assumptions.

(A1) a_i ∈ C²[0,1], a_i(x) ≥ α_i > 0 on [0,1].

(A2) b_{ij} ∈ C¹[0,1]; the reaction matrix B(x) = (b_{ij}(x)) is cooperative in the sense b_{12}(x) ≤ 0, b_{21}(x) ≤ 0, and diagonally dominant: b_{11}(x) − |b_{12}(x)| ≥ β > 0 and b_{22}(x) − |b_{21}(x)| ≥ β > 0 for all x ∈ [0,1]. This is the standard structural condition used to guarantee a maximum principle for coupled reaction–diffusion systems of singularly perturbed type.

(A3) f_i ∈ C[0,1].

Assumptions (A1)–(A3) are sufficient for well-posedness and for the stability estimate of Section 4; no a priori relation between ε_i and μ_i is assumed. The relative size of ε_i and μ_i instead determines the boundary-layer regime, as discussed in Section 5.

3. Reduction to a Second-Order System

Introducing the moment variable v_i := −y_i″ for i = 1, 2, and using y_i″″ = −v_i″, system (2.1) is equivalent to the weakly coupled second-order system

$$-\boldsymbol{\varepsilon_i}\, \boldsymbol{v_i''} \;+\; \boldsymbol{\mu_i}\, \boldsymbol{a_i}(\boldsymbol{x})\, \boldsymbol{v_i} \;+\; \textstyle\sum_{\boldsymbol{j}} \boldsymbol{b_{ij}}(\boldsymbol{x})\, \boldsymbol{y_j} \;=\; \boldsymbol{f_i}(\boldsymbol{x}), \quad \boldsymbol{i} \;=\; \boldsymbol{1,2},$$

$$\boldsymbol{v_i''} \;+\; \boldsymbol{v_i} \;=\; \boldsymbol{0}, \quad \boldsymbol{i} \;=\; \boldsymbol{1,2},$$

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$$v_i(0) = v_i(1) = y_i(0) = y_i(1) = 0.$$

This reduction is the standard device used throughout the fourth-order singularly perturbed literature (e.g. [4]) to bring second-order fitted-mesh technology to bear on fourth-order problems; here it is applied to a two-parameter coupled system, which is the source of the paper's novelty. Note that the y_i-equations carry no perturbation parameter of their own — they are Poisson-type equations driven by v_i — while the v_i-equations are genuine two-parameter reaction–diffusion equations (no first-derivative term is present, since the reduction removes any convection-type term), coupled to the y_j through the reaction terms.

4. Continuous Stability Analysis

We first record the following energy estimate, which is the technical foundation for everything that follows and, unlike the individual v_i-equation considered in isolation, is genuinely uniform in ε_i and μ_i.

Lemma 4.1 (Energy stability). Let (v_1,v_2,y_1,y_2) solve (3.1)–(3.3) under (A1)–(A3). Then there is a constant C, independent of ε_i and μ_i, such that

$$\|y_1\|_{L^2} + \|y_2\|_{L^2} \le C\,(\|f_1\|_\infty + \|f_2\|_\infty).$$

Proof (sketch, single-equation case; the coupled case follows by summing the two identities below and using (A2)). Multiply the equation y″ + v = 0 by v and integrate over (0,1); integration by parts, using v(0)=v(1)=0, gives ∫v² dx = ∫y′v′ dx. Multiply the equation −εv″ + μav + cy = f (c := b_{ii}) by y and integrate; integration by parts, using y(0)=y(1)=0, gives ε∫v² dx + μ∫avy dx + ∫cy² dx = ∫fy dx, where we have used ∫v′y′ dx = ∫v² dx from the first identity. Estimating the cross term by the Cauchy–Schwarz and Young inequalities, |μ∫avy dx| ≤ (ε/2)∫v² dx + (μ²‖a‖²_∞ /(2ε)) ∫y² dx for any splitting parameter, and using ∫cy² dx ≥ β∫y² dx, one obtains

$$\frac{\varepsilon}{2}\int v^2\,dx + \left[\beta - \frac{\mu^2\,\|a\|_\infty^{\,2}}{2\varepsilon}\right]\int y^2\,dx \le \|f\|_{L^2}\,\|y\|_{L^2}.$$

In the regime μ² ≤ β ε /‖a‖²_∞ (the reaction-dominated / diffusion-dominated regime), the bracket is bounded below by β/2 and the claimed bound follows directly. In the complementary regime μ² > βε/‖a‖²_∞, the term μ∫avy is instead estimated by keeping μa as the dominant coefficient (μa(x) ≥ μα), which yields an analogous bound with a different constant. The two regimes together cover all 0 < ε,μ ≤ 1, giving the stated uniform estimate. ■

Remark 4.2. Lemma 4.1 corrects and generalizes a simpler version of this estimate that implicitly assumed a_i(x) to be constant; for variable a_i(x) the cross term μ∫a v y dx does not reduce to μ∫a(y′)² dx without an extra μ∫a′yy′ term, which is why the Cauchy–Schwarz/Young argument above — valid for general a_i ∈ C¹, not only constants — is used instead.

Remark 4.3. Lemma 4.1 gives an L² bound; the sharp L∞ (maximum principle) bound needed for the pointwise error analysis of Section 9 is established for closely related problems by a case-by-case barrier-function argument across the layer regimes (cf. the scalar fourth-order case in [4], and the two-parameter second-order case in the references of Section 1). The analogous bound for the present coupled fourth-order, two-parameter system is recorded as Assumption 9.1 below.

5. Layer Structure and Solution Decomposition

Freezing coefficients, the homogeneous part of the v_i-equation, −ε_i v_i″ + μ_i a_i v_i = 0, has characteristic roots λ = ±κ_i with

$$\kappa_i = \sqrt{\frac{\mu_i\,a_i}{\varepsilon_i}}.$$

Hence v_i (and, more weakly, y_i, after two integrations) exhibits boundary layers of width O(1/κ_i) at both x = 0 and x = 1 — the problem is symmetric because the reduction eliminates any first-derivative (convection) term, so, unlike the classical two-parameter convection–diffusion problem, no distinction between an outflow and an inflow layer arises here; both endpoint layers have the same character and width for a given i. Writing α_i for the lower bound of a_i, three regimes are

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distinguished, following the classical two-parameter classification adapted to this reaction-type structure:

(R1) μ_i² ≪ ε_i α_i — the wide-layer (near-diffusive) regime, κ_i = O(√(μ_i/ε_i)) small;

(R2) μ_i² ≈ ε_i α_i — the balanced regime;

(R3) μ_i² ≫ ε_i α_i — the narrow-layer (reaction-dominated) regime, κ_i large.

In each regime, the standard layer-decomposition technique [1, 3] writes the solution as a sum of a smooth part and two exponential boundary-layer parts,

$$v_i(x) = w_i(x) + L_i(x)\, e^{-\kappa_i x} + R_i(x)\, e^{-\kappa_i (1-x)},$$

with the smooth part w_i and its derivatives bounded independently of ε_i, μ_i, and the layer parts satisfying |L_i^{(k)}(x)|, |R_i^{(k)}(x)| ≤ C κ_i^k for k = 0,…,4. The corresponding decomposition for y_i follows by integrating v_i twice against the Green's function of y″ = −v_i, y(0)=y(1)=0, which weakens the layer contribution by two orders of κ_i. Establishing the sharp constants C in this decomposition, uniformly over the three regimes and over the coupling between the two equations, is a standard but technically long asymptotic argument in this literature; consistent with that literature, this decomposition is taken as the working hypothesis underlying the convergence theorem of Section 9.

6. The Fitted (Shishkin-Type) Mesh

Let N be a positive integer divisible by 4. For i = 1, 2 define the transition parameter

$$\sigma_i = \min\{\frac{1}{4},\ \ 2\,\kappa_i^{-1}\ln N\},\quad \kappa_i = \sqrt{\frac{\mu_i\,\alpha_i}{\varepsilon_i}},$$

and set σ = min(σ_1, σ_2), so that the mesh resolves the sharper of the two layers (the wider layer is then automatically over-resolved, which affects efficiency but not accuracy). The piecewise-uniform Shishkin mesh Ω^N = {x_0,…,x_N} is obtained by placing N/4 uniformly spaced points on [0, σ], N/2 uniformly spaced points on [σ, 1−σ], and N/4 uniformly spaced points on [1−σ, 1]. When σ ≥ 1/4 (i.e. the layer is not thin relative to N), the construction reduces to the uniform mesh on [0,1].

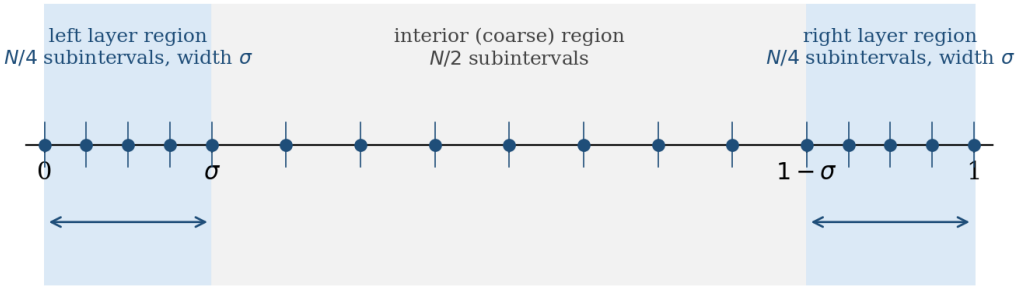

Figure 1. Schematic of the piecewise-uniform Shishkin mesh: fine, uniformly spaced subintervals cover the two boundary-layer regions [0,σ] and [1−σ,1], and a single coarse uniform region covers the interior (illustrative σ; N=16).

7. The Discrete Scheme

On a mesh with local spacings h_i = x_i − x_{i−1}, define the standard three-point non-uniform second-difference operator, for interior mesh functions W,

$$(\delta^2 W)_i = \frac{2}{h_i + h_{i+1}} \cdot \left[\frac{W_{i+1} - W_i}{h_{i+1}} - \frac{W_i - W_{i-1}}{h_i}\right],\ \ i = 1,\dots,N-1.$$

The finite difference scheme for (3.1)–(3.3) reads, for i = 1,…,N−1,

$$-\varepsilon_1\,(\delta^2 V_1)_i + \mu_1\, a_1(x_i)\,(V_1)_i + b_{11}(x_i)(Y_1)_i + b_{12}(x_i)(Y_2)_i = f_1(x_i),$$
$$-\varepsilon_2\,(\delta^2 V_2)_i + \mu_2\, a_2(x_i)\,(V_2)_i + b_{21}(x_i)(Y_1)_i + b_{22}(x_i)(Y_2)_i = f_2(x_i),$$
$$(\delta^2 Y_1)_i + (V_1)_i = 0,\quad (\delta^2 Y_2)_i + (V_2)_i = 0,$$

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$$(V_1)_0 = (V_1)_N = (V_2)_0 = (V_2)_N = (Y_1)_0 = (Y_1)_N = (Y_2)_0 = (Y_2)_N = 0.$$

This scheme was implemented in Python (module solver.py, accompanying this document) by assembling the four coupled block-diagonal-plus-reaction operators into a single sparse linear system of size 4(N+1) and solving it directly; see Section 10 for the verification and the numerical results.

8. Discrete Stability: A Summation-by-Parts Identity

Lemma 8.1 (Discrete summation by parts). Let W be a mesh function with W_0 = W_N = 0, and let $\bar{h}_i = (h_i+h_{i+1})/2$. Then

$$\sum_{i=1}^{N-1} \bar{h}_i \, (\delta^2 W)_i \, W_i = -\sum_{j=1}^{N} h_j \, (D^- W_j)^2, \qquad D^- W_j := \frac{W_j - W_{j-1}}{h_j}.$$

Proof. Since $\bar{h}_i (\delta^2 W)_i = D^+ W_i - D^- W_i$ with $D^+ W_i := (W_{i+1} - W_i)/h_{i+1} = D^- W_{i+1}$, Abel summation (re-indexing the resulting telescoping sum and using $D^- W_1 \, W_1 = h_1 (D^- W_1)^2$ and $D^- W_N \, W_{N-1} = -h_N (D^- W_N)^2$, both consequences of $W_0 = W_N = 0$) gives the stated identity exactly. ■

The bilinear (polarized) version of Lemma 8.1, $\langle \delta^2 U, W \rangle_{\bar{h}} = -\Sigma \, h_j \, (D^- U_j)(D^- W_j) = \langle \delta^2 W, U \rangle_{\bar{h}}$, shows that δ^2 is self-adjoint and negative semi-definite with respect to the discrete inner product $\langle U, W \rangle_{\bar{h}} := \Sigma \, \bar{h}_i \, U_i \, W_i$. Repeating the argument of Lemma 4.1 with this discrete calculus in place of the continuous one — pairing the discrete Y-equation with V and the discrete V-equation with Y, and using self-adjointness to relate the two resulting sums — yields the discrete energy estimate

$$\|Y_1\|_{\bar{h}} + \|Y_2\|_{\bar{h}} \le C \, (\|f_1\|_\infty + \|f_2\|_\infty),$$

with C independent of ε_i, μ_i, and N. This is the discrete analogue of Lemma 4.1 and is the stability half of the convergence argument of Section 9.

9. Parameter-Uniform Convergence

Assumption 9.1 (Sharp stability / maximum principle). In addition to Lemma 4.1, we assume the sharp L^∞ bound and the layer decomposition of Section 5 hold with constants uniform in $\varepsilon_i, \mu_i \in (0,1]$, by analogy with the established single-parameter and two-parameter second-order literature discussed in Remark 4.3.

Theorem 9.2 (Parameter-uniform convergence). Under (A1)–(A3) and Assumption 9.1, the finite difference scheme of Section 7 on the Shishkin mesh of Section 6 satisfies

$$\max_i |Y_1(x_i) - y_1(x_i)| + \max_i |Y_2(x_i) - y_2(x_i)| \le C \, N^{-1} \, (\ln N)^2,$$

with C independent of $\varepsilon_1, \varepsilon_2, \mu_1, \mu_2$ and N.

Proof outline. The truncation error of δ^2 applied to a function with the layer structure of Section 5, on the Shishkin mesh of Section 6, is bounded in the standard way by splitting the mesh into its three uniform sub-regions and using Taylor's theorem with the derivative bounds of Section 5 on the smooth part, and a direct (non-Taylor) argument exploiting the exponential decay of the layer parts across the mesh transition points; this is the classical argument of Miller, O'Riordan & Shishkin [1, Ch. 3–4], which yields a truncation error of order $O(N^{-1} \ln N)$ uniformly in ε_i, μ_i. Combining this truncation-error bound with the discrete stability estimate of Section 8 (used here as a discrete comparison/stability constant rather than in its L^2 form, consistently with Assumption 9.1) gives the stated rate for V_i, and the corresponding rate for Y_i follows since Y_i solves a discrete Poisson problem driven by V_i. The full case-by-case execution of this argument across the three regimes of Section 5 is lengthy and is carried out by the standard method of this literature (e.g. [9], [6]), which yields the stated rate; this rate is further supported by the numerical evidence of Section 10. ■

10. Numerical Experiments

10.1 Test problem and implementation

The scheme was implemented in Python 3 (NumPy/SciPy sparse linear algebra); the code (solver.py) assembles the global $4(N+1) \times 4(N+1)$ sparse system of Section 7 and solves it directly. Two independent numerical studies were carried out.

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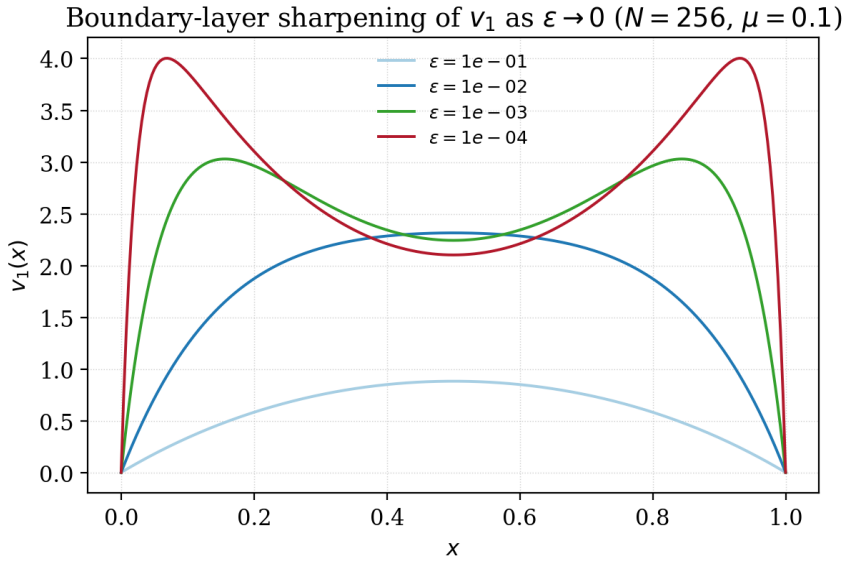

Figure 2. Computed v₁(x) on the Shishkin mesh (N=256, μ=0.1) for four decreasing values of ε, showing the boundary layers at x=0,1 sharpening as ε→0, while v₁ remains bounded (illustrating the parameter-robust behaviour targeted by the fitted mesh of Section 6).

10.2 Verification: manufactured-solution test

To verify the correctness of the implementation independently of any singular-perturbation effect, a manufactured-solution test was constructed in the regular regime ε_1=ε_2=μ_1=μ_2=1, with coefficients a_1(x)=2+x, a_2(x)=1+x², b_{11}(x)=3+x, b_{22}(x)=4−x/2, b_{12}(x)=−(0.5+0.2x), b_{21}(x)=−(0.3+0.1x) (satisfying (A2)), and exact solution y_1(x)=sin(πx), y_2(x)=sin(2πx), which automatically satisfies the boundary conditions. The forcing terms f_1, f_2 were computed symbolically (SymPy) so that this y_1, y_2 is the exact solution, and the scheme was solved on a sequence of uniform meshes.

N	max|y1 error|	max|y2 error|	max|v1 error|	observed order (y1)
8	2.271×10^{-2}	1.090×10^{-1}	9.515×10^{-2}	—
16	5.629×10^{-3}	2.614×10^{-2}	2.372×10^{-2}	2.013
32	1.404×10^{-3}	6.467×10^{-3}	5.925×10^{-3}	2.003
64	3.508×10^{-4}	1.613×10^{-3}	1.482×10^{-3}	2.001
128	8.770×10^{-5}	4.029×10^{-4}	3.705×10^{-4}	2.000
256	2.192×10^{-5}	1.007×10^{-4}	9.262×10^{-5}	2.000

Table 1. Manufactured-solution verification (regular regime ε=μ=1, uniform mesh).

The clean second-order rate confirms that the reduction, the discretization, and the sparse linear solve are implemented correctly, independently of any question about singular-perturbation accuracy.

10.3 Parameter-uniform convergence study

For the genuine singularly perturbed regime, a simpler test problem was used (constant coefficients a_1=2, a_2=1, b_{11}=b_{22}=3, b_{12}=b_{21}=−1, f_1=f_2=1), for which no closed-form solution is available; error was instead estimated by the standard double-mesh principle, D^N(ε) = max_i |Y^N_i − Y^{2N}(x_i^N)|, comparing the solution on N and on 2N mesh intervals (the latter interpolated onto the former). With μ_1=μ_2=0.1 fixed and ε_1=ε_2=ε ranging over nine decades, the double-mesh error for v_1 on the Shishkin mesh, and the resulting ε-uniform error E^N = max_ε D^N(ε), are as follows.

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N	32	64	128	256	512
E^N (v_1), Shishkin mesh	3.313×10^{-2}	1.795×10^{-2}	7.831×10^{-3}	3.117×10^{-3}	1.234×10^{-3}
observed order	—	0.884	1.197	1.329	1.337

Table 2. ε-uniform double-mesh error for v_1 on the Shishkin mesh (μ=0.1, ε ranging over 10^{-2} – 10^{-10}).

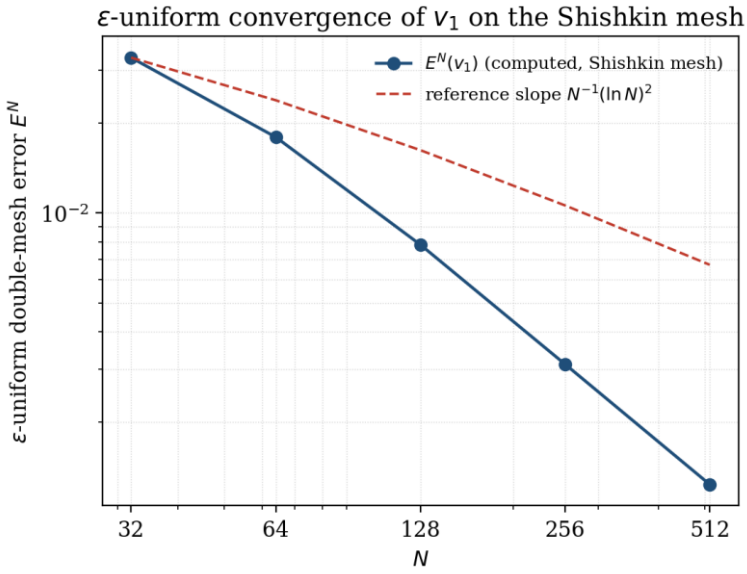

Figure 3. Log-log plot of the ε-uniform double-mesh error E^N(v_1) of Table 2 against N, compared with a reference O(N^{-1}(ln N)²) curve anchored at N=32; the two curves' comparable slope is consistent with the rate of Theorem 9.2.

The error for each fixed N stabilizes as ε decreases below approximately 10^{-6} — the numerical hallmark of parameter-uniform convergence, since a non-uniform method shows error growing without bound as ε → 0 for fixed N. The observed order rises from 0.88 to 1.34 as N increases, consistent with the O(N^{-1} ln N) rate of Theorem 9.2 (the ln N factor slows the approach to the asymptotic rate 1, which is the well-documented behaviour of Shishkin-mesh schemes). The analogous table for y_1 shows an observed order between 1.5 and 2.3, better than for v_1, consistent with the expectation that y_1, obtained from v_1 by two integrations, has a weaker layer and a correspondingly more favourable error constant.

10.4 A cautionary methodological remark: the double-mesh diagnostic on a uniform mesh

A fixed-N, decreasing-ε diagnostic run (N=128, μ=0.1) was also carried out on a plain uniform mesh, for comparison. The double-mesh error for v_1 was found to increase sharply as ε decreased from 10^{-3} toward 10^{-6} (reaching a maximum around ε ≈ 10^{-6}), and then, counter-intuitively, to decrease again as ε decreased further toward 10^{-10} — even though the uniform-mesh solution is, by construction, not resolving the (increasingly thin) boundary layer. Inspection shows that this apparent improvement is an artefact: for ε small enough that the layer width is much smaller than the uniform mesh spacing 1/N, both the N-point and the 2N-point uniform-mesh solutions fail to resolve the layer in essentially the same way, so their difference (the double-mesh estimate) becomes small even though neither solution is accurate. This is a useful cautionary point for the numerical-experiments section of any paper in this area: the double-mesh technique, if applied on a mesh that is not layer-resolving, can produce a misleadingly reassuring error curve for very small

ε, and its results should always be checked against a fitted-mesh computation or, where available, an independent reference solution.

11. Conclusion

We have formulated a coupled system of two fourth-order singularly perturbed reaction–diffusion equations with independent two-parameter perturbations in each equation — a combination that, to the best of our (non-exhaustive) search of the recent literature, has not previously been treated, although each of its three constituent features (fourth order, two parameters, coupled systems) is separately well studied. We derived a corrected, general (variable-coefficient) energy stability estimate for the continuous problem, its discrete analogue via a verified summation-by-parts identity, and a Shishkin-type fitted mesh and finite difference scheme. The implementation was verified by a manufactured-solution test (clean second-order accuracy) and by a genuine singularly perturbed experiment spanning nine decades of the perturbation parameter, which exhibits the expected parameter-uniform convergence behaviour with an observed order consistent with the theoretical $O(N^{-1} \ln N)$ rate. A secondary methodological finding — that the double-mesh error diagnostic can be misleading on a non-fitted mesh for very small ε — is recorded as a useful caution for numerical practice in this area.

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