



COEFFICIENT ESTIMATES FOR SUBFAMILIES OF BI-UNIVALENT FUNCTIONS INVOLVING EULER POLYNOMIALS

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Abstract. The present research introduces a novel subclass of bi-univalent functions that incorporates Euler polynomials. This study obtains the bounds for coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series. Moreover, the function's Fekete- Szegő $|a_3 - M a_2^2|$ problem is also described. Ultimately, a number of corollaries of the most significant theorems offer fascinating new information, and the related research highlights how essential the findings are.

Keywords: bi-univalent; Euler polynomials; Fekete- Szegő.

1. Introduction

In mathematics, univalent function theory is one of the most interesting different areas in complex analytic. Finding complex-valued functions' geometric properties is the main goal of this topic. This region aims at establishing limits for the coefficients of functions that are members of some subclasses of injective and regular functions. 'Univalent' or 'Schlicht' denotes a complex-valued function $f: \Theta \subset \mathbb{C} \rightarrow \mathbb{C}$ that that doesn't assume an identical value more than once.

Especially, the significance of the Riemann mapping theorem has increased interest in univalent functions [1], [2]. The Bieberbach hypothesis is one of the most significant problems of the 20th century [3]. It claims that the n -th coefficients of an analytic and univalent function are limited. Many mathematicians in 19th century were interested in the Bieberbach conjecture. During that period, a lot of effort was done on solving the mentioned problem for many regular subclasses and univalent functions; nevertheless, the idiosyncrasies of these function classes were also explored. Moreover, recent papers dealt with the Fekete-Szegö problem [4], [5], [3], [6], [7] concerning these subclasses from certain polynomials based on their coefficients $|a_2|$ and $|a_3|$.

Suppose \check{A} is the class of functions that are analytic inside the open unit disc $U = \{z: z \in \mathbb{C}: |z| < 1\}$ and possess a Taylor-Maclaurin series expansion expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (z \in U) \quad (1.1)$$

Additionally, functions in \mathcal{A} are normalized by the following conditions:

$$f(0) = f'(0) - 1 = 0 .$$

Allow \check{X} to consist of functions which are univalent in U , such that all function $f \in \check{X}$ has an inverse f^{-1} , that fulfils this requirement [1]:

$$f^{-1}(f(z)) = z, (z \in U) \text{ and } f(f^{-1}(\omega)) = \omega, (|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}),$$

where

$$\begin{aligned} g(\omega) &= f^{-1}(\omega) \\ &= \omega - a_2 \omega^2 + (2a_2^2 - a_3) \omega^3 - (5a_2^3 - 5a_2 a_3 + a_4) \omega^4 + \dots \end{aligned} \quad (1.2)$$

Let \check{Y} denote the class of bi-univalent functions in the domain U . f and f^{-1} are univalent over U , then f is bi- univalent over U [8].

suppose f and g and are analytic in U , we use the notation f is subordinate to g , if there is a Schwarz function $\check{\omega}(z)$ which is analytic in U with $\check{\omega}(0) = 0$ and $|\check{\omega}(z)| < 1$, where $f(z) = g(\check{\omega}(z))$, $(z \in U)$. For further details regarding the subordination concept, it is advisable to refer to the publications [9], [10], [11], [12], [13], [14].

Furthermore, we will have the following equivalence provided that the function g is univalent in U

$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For understanding the nature of complex functions and its geometrical structure, Euler polynomials were introduced by Leonhard Euler in the early 17 th century. They are important in the theory of conformal maps that locally maintain angles in the theory of geometric functions. They are widely used in several geometric function theory domains as Schwarz Christoffel mappings, Riemann surface theory, univalent functions and others. Explanations such as this help to demystify the relationship between stabs functions polynomials and geometry transformations.

The purpose of the present paper is the study of such special function and Euler polynomial. We attempt to define a new and large class of bi-univalent functions.

Euler's polynomials are often described through the generating function $\bar{W}_k(\mathcal{B})$ [15], [16], [17].

$$K(\mathcal{B}, z) = \frac{2e^{z\mathcal{B}}}{e^z + 1} = \sum_{k=0}^{\infty} \bar{W}_k(\mathcal{B}) \frac{z^k}{k!}, \quad \left(\frac{1}{2} < \mathcal{B} \leq 1; |z| < \pi\right).$$

A definitive expression for $\bar{W}_k(\mathcal{B})$ is given by

$$\bar{W}_j(\mathcal{B}) = \sum_{i=0}^j \frac{1}{2^i} \sum_{u=0}^i (-1)^u \binom{i}{u} (\mathcal{B} + u)^j \quad (1.3)$$

Now, the function $\bar{W}_j(\mathcal{B})$ in terms of \bar{W}_u can be obtained from the above equation as:

$$\bar{W}_i(\mathcal{B}) = \sum_{u=0}^i \frac{\bar{W}_u}{2^u} \binom{i}{u} \left(\mathcal{B} - \frac{1}{2}\right)^{i-u}.$$

Listed below are the beginning values of Euler polynomial:

$$\begin{aligned} \bar{W}_0(\mathcal{B}) &= 1, \\ \bar{W}_1(\mathcal{B}) &= \frac{(2\mathcal{B} - 1)}{2}, \\ \bar{W}_2(\mathcal{B}) &= \mathcal{B}^2 - \mathcal{B}, \end{aligned} \quad (1.4)$$

$$\bar{W}_3(\mathcal{B}) = \frac{(4\mathcal{B}^3 - 6\mathcal{B}^2 + 1)}{4}, \quad (1.6)$$

$$\bar{W}_4(\mathcal{B}) = \mathcal{B}^4 - 2\mathcal{B}^3 + \mathcal{B}. \quad (1.7)$$

Many recent studies have investigated geometric function theory, focusing on coefficient estimates. Multiple diverse groups within the class were established, and approximate values for the coefficients $|a_2|$ and $|a_3|$, as well as the Taylor-Maclaurin series expansion (1), were derived from sources [18], [19], [20], [21], [22], [23].

This paper presents a novel subclass $\mathcal{I}(\mu, \lambda, \mathcal{B})$ of \check{Y} associated with the Euler polynomials and establishes bounds for the $|a_2|$ and $|a_3|$ Taylor-Maclaurin coefficients and Fekete- Szegő issues. Moreover, the investigation uncovers other innovative discoveries.

2. Definitions and Examples

We define the new subclass $\mathcal{I}(\mu, \lambda, \mathcal{B})$ associated with Euler polynomials at the beginning of this section, and we will discuss it in the next section.

Definition 2.1. A function $f \in \check{Y}$ defined by (1.1), it classifies as belonging to the subclass $\mathcal{I}(\mu, \lambda, \mathcal{B})$ if it meets the following criteria for subordinations:

$$\frac{z(f'(z))^\mu}{(1-\lambda)z + \lambda f(z)} + \frac{zf''(z)}{f'(z)} + 1 < K(\mathcal{B}, z) + 1 = \sum_{k=0}^{\infty} \bar{W}_k(\mathcal{B}) \frac{z^k}{k!} + 1 \quad (2.1)$$

and

$$\frac{\omega(g'(\omega))^\mu}{(1-\lambda)\omega + \lambda g(\omega)} + \frac{\omega g''(\omega)}{g'(\omega)} + 1 < K(\mathcal{B}, \omega) + 1 = \sum_{k=0}^{\infty} \bar{W}_k(\mathcal{B}) \frac{\omega^k}{k!} + 1 \quad (2.2)$$

where $\lambda \in [0,1]$; $\mu \geq 1$, $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$.

Example 2.1. A function $f \in \check{H}$ defined by (1.1) classified as belonging to the subclass $\mathcal{I}(1, \lambda, \mathcal{B})$ if it meets the following criteria for subordinations:

$$\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} + \frac{zf''(z)}{f'(z)} + 1 < K(\mathcal{B}, z) + 1 \quad (2.3)$$

and

$$\frac{\omega g'(\omega)}{(1-\lambda)\omega + \lambda g(\omega)} + \frac{\omega g''(\omega)}{g'(\omega)} + 1 < K(\mathcal{B}, \omega) + 1 \quad (2.4)$$

where $\lambda \in [0,1]$; $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$.

Example 2.2. A function $f \in \check{H}$ defined by (1.1) classified as belonging to the subclass $\mathcal{I}(1,1, \mathcal{B})$ if it meets the following criteria for subordinations:

$$\frac{zf'(z)}{\lambda f(z)} + \frac{zf''(z)}{f'(z)} + 1 < K(\mathcal{B}, z) + 1 \quad (2.5)$$

and

$$\frac{\omega g'(\omega)}{\lambda g(\omega)} + \frac{\omega g''(\omega)}{g'(\omega)} + 1 < K(\mathcal{B}, \omega) + 1 \quad (2.6)$$

where $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$.

Example 2.3. A function $f \in \check{H}$ defined by (1.1) classified as belonging to the subclass $\mathcal{I}(1,0, \mathcal{B})$ if it meets the following criteria for subordinations:

$$f'(z) + \frac{zf''(z)}{f'(z)} + 1 < K(\mathcal{B}, z) + 1 \quad (2.7)$$

and

$$g'(\omega) + \frac{\omega g''(\omega)}{g'(\omega)} + 1 < K(\mathcal{B}, \omega) + 1 \quad (2.8)$$

where $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$.

Lemma 2.1. [2] Let Ψ denote the set of all analytic functions. permit $d \in \Psi$, in which

$$d(z) = 1 + \sum_{i=1}^{\infty} d_i z^i, (z \in \mathbb{U}).$$

with $\text{Re}(d(z)) > 0$ and $d(0) = 1$. Then $|d_i| \leq 2$, $i = 1, 2, \dots$

3. Bounds of the class $\mathcal{I}(\mu, \lambda, \mathcal{B})$

We provide the coefficient estimates for a function $f \in U$ and address the Fekete-Szegö [17] for the relevant class $\mathcal{I}(\mu, \lambda, \mathcal{B})$.

Theorem 3.1. Let f given by (1) be in $\mathcal{H}(\mu, \lambda, \mathcal{B})$, then

$$|a_2| \leq \sqrt{\phi(\mu, \lambda, \mathcal{B})},$$

and

$$|a_3| \leq \frac{(2\mathcal{B} - 1)^2}{4(2\mu - \lambda + 2)^2} + \frac{(2\mathcal{B} - 1)}{2(6 - \lambda + 2\mu)},$$

where

$$\phi(\mu, \lambda, \mathcal{B}) = \sqrt{\frac{(2\mathcal{B} - 1)^3}{2((\mu - \lambda)(2\mu + 1) + \lambda^2 + 2)(2\mathcal{B} - 1)^2 - 4(2\mu - \lambda + 2)^2} (\mathcal{B}^2 - 3\mathcal{B} + 1)}$$

where $\lambda \in [0, 1]$; $\mu \geq 1$, $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$.

Proof: Given that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{H}(\mu, \lambda, \mathcal{B})$, where $\lambda \in [0, 1]$; $\mu \geq 1$, $\mathcal{B} \in (\frac{1}{2}, 1]$; and $g = f^{-1}$, it follows from (2.1) and (2.2) we have

$$\frac{z(f'(z))^\mu}{(1 - \lambda)z + \lambda f(z)} + \frac{zf''(z)}{f'(z)} + 1 < K(\mathcal{B}, \tau_1(z)) + 1 \quad (3.1)$$

and

$$\frac{\omega(g'(\omega))^\mu}{(1 - \lambda)\omega + \lambda g(\omega)} + \frac{\omega g''(\omega)}{g'(\omega)} + 1 < K(\mathcal{B}, \tau_2(\omega)) + 1 \quad (3.2)$$

where the functions τ_1, τ_2 defined as

$$\tau_1, \tau_2: U \rightarrow U, \text{ with } \tau_1(0) = \tau_2(0) = 0,$$

and $|\tau_1(z)|, |\tau_2(\omega)| < 1$ for all $z, \omega \in U$.

So, we can define $\rho, \sigma \in \Omega$ to be of the form

$$\rho(z) = \frac{\tau_1(z) + 1}{1 - \tau_1(z)} = 1 + \rho_1 z + \rho_2 z^2 + \rho_3 z^3 + \dots, |\rho_k| \leq 2, \text{ for all } k \in \mathbb{N},$$

This leads to

$$\begin{aligned} \tau_1(z) &= \frac{\rho(z) - 1}{1 + \rho(z)} \\ &= \frac{\rho_1}{2} z + \left(\frac{\rho_2}{2} - \frac{\rho_1^2}{4} \right) z^2 + \frac{1}{2} \left(\rho_3 - \rho_1 \rho_2 + \frac{\rho_1^3}{4} \right) z^3 + \dots \end{aligned} \quad (3.3)$$

and

$$\sigma(\omega) = \frac{\tau_2(\omega) + 1}{1 - \tau_2(\omega)} = 1 + \sigma_1 \omega + \sigma_2 \omega^2 + \sigma_3 \omega^3 + \dots, |\sigma_k| \leq 2, \text{ for all } k \in \mathbb{N},$$

This leads to

$$\begin{aligned} \tau_2(\omega) &= \frac{\sigma(\omega) - 1}{1 + \sigma(\omega)} \\ &= \frac{\sigma_1}{2} \omega + \left(\frac{\sigma_2}{2} - \frac{\sigma_1^2}{4} \right) \omega^2 + \frac{1}{2} \left(\sigma_3 - \sigma_1 \sigma_2 + \frac{\sigma_1^3}{4} \right) \omega^3 + \dots \end{aligned} \quad (3.4)$$

Using (3.3) and (3.4), we obtain

$$\begin{aligned} \kappa(\mathcal{B}, \tau_1(z)) + 1 = 1 + \bar{W}_0(\mathcal{B}) + \frac{\bar{W}_1(\mathcal{B})}{2} \rho_1 z + \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\rho_2 - \frac{\rho_1^2}{2} \right) + \frac{\bar{W}_2(\mathcal{B})}{8} \rho_1^2 \right] z^2 + \\ \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\rho_3 - \rho_1 \rho_2 + \frac{\rho_1^3}{4} \right) + \frac{\bar{W}_2(\mathcal{B})}{4} \left(\rho_1 \rho_2 - \frac{\rho_1^3}{2} \right) + \frac{\bar{W}_3(\mathcal{B})}{48} \rho_1^3 \right] z^3 \\ + \dots \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \kappa(\mathcal{B}, \tau_2(\omega)) + 1 = 1 + \bar{W}_0(\mathcal{B}) + \frac{\bar{W}_1(\mathcal{B})}{2} \sigma_1 \omega + \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\sigma_2 - \frac{\sigma_1^2}{2} \right) + \frac{\bar{W}_2(\mathcal{B})}{8} \sigma_1^2 \right] \omega^2 + \\ \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\sigma_3 - \sigma_1 \sigma_2 + \frac{\sigma_1^3}{4} \right) + \frac{\bar{W}_2(\mathcal{B})}{4} \left(\sigma_1 \sigma_2 - \frac{\sigma_1^3}{2} \right) + \frac{\bar{W}_3(\mathcal{B})}{48} \sigma_1^3 \right] \omega^3 \\ + \dots \end{aligned} \quad (3.6)$$

from (3.1) and (3.2) and the previous two equations, we have

$$\begin{aligned} 2 + (2\mu - \lambda + 2)a_2 z + \left[\left(\frac{2\mu^2 - 2\mu(\lambda + 1)}{\lambda^2 - 4} \right) a_2^2 + (6 - (\lambda - 3\mu))a_3 \right] z^2 \\ + \left[\left(\frac{2\lambda^2 - 5\lambda\mu + 6\mu(\mu - 1) - 18}{6\mu(\mu - 1) - 18} \right) a_2 a_3 + \left(8 - \frac{1}{3} \left(\frac{3\lambda^3 - 6\lambda^2\mu + 6\lambda\mu(\mu - 1)}{-4\mu(\mu^2 - 3\mu + 2)} \right) a_2^3 \right) + (4\mu - \lambda + 12)a_4 \right] z^3 \dots \\ = 2 + \frac{\bar{W}_1(\mathcal{B})}{2} \rho_1 z + \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\rho_2 - \frac{\rho_1^2}{2} \right) + \frac{\bar{W}_2(\mathcal{B})}{8} \rho_1^2 \right] z^2 \\ + \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\rho_3 - \rho_1 \rho_2 + \frac{\rho_1^3}{4} \right) + \frac{\bar{W}_2(\mathcal{B})}{4} \left(\rho_1 \rho_2 - \frac{\rho_1^3}{2} \right) + \frac{\bar{W}_3(\mathcal{B})}{48} \rho_1^3 \right] z^3 \\ + \dots \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} 2 - [2\mu - \lambda + 2]a_2 \omega + \left[\left(\frac{2\mu^2 - (2\mu - \lambda)(2 - \lambda)}{+8} \right) a_2^2 + (\lambda - 3\mu - 6)a_3 \right] \omega^2 \\ + \left[\left(\frac{2\lambda^2 - 5\lambda\mu(\mu + 1) + 2\mu(3\mu + 7) + 42}{-4\mu(\mu^2 + 6\mu + 8)} \right) a_2 a_3 + \left(\frac{1}{3} \left(\frac{3\lambda^3 - 6\lambda^2(\mu + 2) + 3\lambda(2\mu^2 + 8\mu + 5)}{-4\mu(\mu^2 + 6\mu + 8)} \right) - 32 \right) a_2^3 \right] \omega^3 \dots \\ - (4\mu - \lambda + 12)a_4 \end{aligned}$$

$$\begin{aligned}
&= 2 + \frac{\bar{W}_1(\mathcal{B})}{2} \sigma_1 \omega + \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\sigma_2 - \frac{\sigma_1^2}{2} \right) + \frac{\bar{W}_2(\mathcal{B})}{8} \sigma_1^2 \right] \omega^2 \\
&+ \left[\frac{\bar{W}_1(\mathcal{B})}{2} \left(\sigma_3 - \sigma_1 \sigma_2 + \frac{\sigma_1^3}{4} \right) + \frac{\bar{W}_2(\mathcal{B})}{4} \left(\sigma_1 \sigma_2 - \frac{\sigma_1^3}{2} \right) \right. \\
&\quad \left. + \frac{\bar{W}_3(\mathcal{B})}{48} \sigma_1^3 \right] \omega^3 \dots \tag{3.8}
\end{aligned}$$

By equating the coefficients in (3.7), (3.8), we obtain

$$(2\mu - \lambda + 2)a_2 = \frac{\bar{W}_1(\mathcal{B})}{2} \rho_1, \tag{3.9}$$

$$\begin{aligned}
&\left(\frac{2\mu^2 - 2\mu(\lambda + 1)}{+\lambda^2 - 4} \right) a_2^2 + \left(\frac{6 -}{(\lambda - 3\mu)} \right) a_3 \\
&= \frac{\bar{W}_1(\mathcal{B})}{2} \left(\rho_2 - \frac{\rho_1^2}{2} \right) + \frac{\bar{W}_2(\mathcal{B})}{8} \rho_1^2 \tag{3.10}
\end{aligned}$$

Also,

$$-[2\mu - \lambda + 2]a_2 = \frac{\bar{W}_1(\mathcal{B})}{2} \sigma_1 \tag{3.11}$$

$$\begin{aligned}
&\left(\frac{2\mu^2 - (2\mu - \lambda)}{(2 - \lambda) + 8} \right) a_2^2 + \left(\frac{\lambda - 3\mu}{-6} \right) a_3 \\
&= \frac{\bar{W}_1(\mathcal{B})}{2} \left(\sigma_2 - \frac{\sigma_1^2}{2} \right) + \frac{\bar{W}_2(\mathcal{B})}{8} \sigma_1^2 \tag{3.12}
\end{aligned}$$

Upon adding (3.9) to (3.11) and some simplification, we get

$$\rho_1 = -\sigma_1 \tag{3.13}$$

and

$$\rho_1^2 + \sigma_1^2 = \frac{8(2\mu - \lambda + 2)^2 a_2^2}{(\bar{W}_1(\mathcal{B}))^2} \tag{3.14}$$

$$\begin{aligned}
&= \frac{(\bar{W}_1(\mathcal{B}))^2 (\rho_1^2 + \sigma_1^2)}{8(2\mu - \lambda + 2)^2} \tag{3.15}
\end{aligned}$$

More our, adding (3.10) to (3.12), the result

$$2 \left(\frac{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} \right) a_2^2 = \frac{\bar{W}_1(\mathcal{B})}{2} (\rho_2 + \sigma_2) + (\rho_1^2 + \sigma_1^2) \left(\frac{\bar{W}_2(\mathcal{B})}{8} - \frac{\bar{W}_1(\mathcal{B})}{4} \right)$$

Since $\rho_1^2 = \sigma_1^2$, we get

$$\begin{aligned}
&16 \left(\frac{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} \right) a_2^2 \\
&= 4\bar{W}_1(\mathcal{B})(\rho_2 + \sigma_2) + 2\rho_1^2 \left(\frac{\bar{W}_2(\mathcal{B})}{2\bar{W}_1(\mathcal{B})} - \right) \tag{3.16}
\end{aligned}$$

From (3.13) and (3.14), we obtain

$$8(2\mu - \lambda + 2)^2 a_2^2 = 2\rho_1^2 (\bar{W}_1(\mathcal{B}))^2$$

So,

$$= \frac{4(2\mu - \lambda + 2)^2 a_2^2}{(\bar{W}_1(\mathcal{B}))^2} \rho_1^2 \quad (3.17)$$

By replacing ρ_1^2 in (3.16), we obtain

$$\left(16 \binom{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} \right) a_2^2 - \frac{8(2\mu - \lambda + 2)^2 a_2^2}{(\bar{W}_1(\mathcal{B}))^2} (\bar{W}_2(\mathcal{B}) - 2\bar{W}_1(\mathcal{B})) = 4\bar{W}_1(\mathcal{B})(\rho_2 + \sigma_2)$$

Consequently,

$$a_2^2 = \frac{(\bar{W}_1(\mathcal{B}))^3 (\rho_2 + \sigma_2)}{\left[4 \binom{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} (\bar{W}_1(\mathcal{B}))^2 - 2 \binom{\bar{W}_2(\mathcal{B}) - 2\bar{W}_1(\mathcal{B})}{2\bar{W}_1(\mathcal{B})} (2\mu - \lambda + 2)^2 \right]} \quad (3.18)$$

Therefore,

$$|a_2|^2 \leq \frac{(\bar{W}_1(\mathcal{B}))^3 (|\rho_2| + |\sigma_2|)}{\left[\left[4 \binom{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} (\bar{W}_1(\mathcal{B}))^2 - 2 (\bar{W}_2(\mathcal{B}) - 2\bar{W}_1(\mathcal{B})) (2\mu - \lambda + 2)^2 \right] \right]}$$

Applying Lemma 2.1 and (2.4), we have:

$$|a_2|^2 \leq \frac{(2\mathcal{B} - 1)^3}{\left[\left[2 \binom{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} (2\mathcal{B} - 1)^2 - 4 \binom{\mathcal{B}^2 - 3\mathcal{B}}{+1} \binom{2\mu - \lambda}{+2} \right] \right]}$$

So,

$$|a_2| \leq \sqrt{\frac{(2\mathcal{B} - 1)^3}{\left[\left[2 \binom{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} (2\mathcal{B} - 1)^2 - 4 \binom{\mathcal{B}^2 - 3\mathcal{B}}{+1} \binom{2\mu - \lambda}{+2} \right] \right]}} = \sqrt{\mathfrak{D}(\mu, \lambda, \mathcal{B})}$$

where

$$\mathfrak{D}(\mu, \lambda, \mathcal{B}) = \frac{(2\mathcal{B} - 1)^3}{\left[\left[2 \binom{(\mu - \lambda)(2\mu + 1)}{+\lambda^2 + 2} (2\mathcal{B} - 1)^2 - 4 (\mathcal{B}^2 - 3\mathcal{B} + 1) (2\mu - \lambda + 2)^2 \right] \right]}$$

To determine the estimates on $|a_3|$, by subtracting (3.12) from (3.10) and in view of (3.13), we obtain.

$$= a_2^2 + \frac{\bar{W}_1(\mathcal{B})(\rho_2 - \sigma_2)}{4(6 - \lambda + 3\mu)} a_3 \quad (3.19)$$

This results in the following inequality

$$\leq |a_2|^2 + \frac{\bar{W}_1(\mathcal{B})(|\rho_2| + |\sigma_2|)}{4(6 - \lambda + 3\mu)} |a_3| \quad (3.20)$$

Replacing $|a_2|^2$ by (3.15) and by (3.13), we obtain

$$|a_3| \leq \frac{(\bar{W}_1(\mathcal{B}))^2 |\rho_1|^2}{4(2\mu - \lambda + 2)^2} + \frac{\bar{W}_1(\mathcal{B})(|\rho_2| + |\sigma_2|)}{4(6 - \lambda + 3\mu)}$$

Applying Lemma 2.1 and (2.4), we get:

$$\leq \frac{(2\mathcal{B} - 1)^2}{4(2\mu - \lambda + 2)^2} + \frac{|a_3|}{2(6 - \lambda + 3\mu)} \quad (3.21)$$

The proof of Theorem 3.1 is completed. ■

4. Fekete- Szegő Functional Estimations

Now, Finding the value of a_2^2 and a_3 helps establish the Fekete- Szegő [21] inequality for functions belonging to $\mathcal{H}(\mu, \lambda, \mathcal{B})$.

Lemma 4.1. let \mathcal{U} and \mathcal{J} be real numbers. Let m and n be complex numbers. If $|m| < \mathcal{R}$ and $|n| < \mathcal{R}$, then

$$|(\mathcal{U} + \mathcal{J})m + (\mathcal{U} - \mathcal{J})n| \leq \begin{cases} 2\mathcal{R}|\mathcal{U}| & \text{if } |\mathcal{U}| \geq |\mathcal{J}| \\ 2\mathcal{R}|\mathcal{J}| & \text{if } |\mathcal{U}| \leq |\mathcal{J}| \end{cases}$$

Theorem 4.1. Assume f belongs to the specified class $\mathcal{H}(\mu, \lambda, \mathcal{B})$ that defined as (1.1). Then

$$|a_3 - M_1 a_2^2| \leq \begin{cases} \frac{(2\mathcal{B} - 1)}{2(6 - \lambda + 3\mu)} & \text{if } 0 \leq |\hbar(M_1)| \leq \frac{1}{4(6 - \lambda + 3\mu)} \\ 2(2\mathcal{B} - 1)|\hbar(M_1)| & \text{if } |\hbar(M_1)| \geq \frac{1}{4(6 - \lambda + 3\mu)} \end{cases}$$

where

$$\hbar(M_1) = \frac{(\bar{W}_1(\mathcal{B}))^2(1 - M_1)}{\left[4 \left(\frac{(\mu - \lambda)(2\mu + 1)}{\lambda^2 + 2} \right) (\bar{W}_1(\mathcal{B}))^2 - 2 \left(\frac{\bar{W}_2(\mathcal{B})}{2\bar{W}_1(\mathcal{B})} \right) (2\mu - \lambda + 2)^2 \right]}$$

(where $\lambda \in [0, 1]$; $\mu \geq 1$, $\mathcal{B} \in \left(\frac{1}{2}, 1\right]$; $z, \omega \in U$, M_1 be a real number and $g = f^{-1}$).

Proof: Based on (3.18) and (3.19), we obtain

$$\begin{aligned} a_3 - M_1 a_2^2 &= a_2^2 + \frac{\bar{W}_1(\mathcal{B})(\rho_2 - \sigma_2)}{4(6 - \lambda + 3\mu)} - M_1 a_2^2 \\ &= (1 - M_1) a_2^2 + \frac{\bar{W}_1(\mathcal{B})(\rho_2 - \sigma_2)}{4(6 - \lambda + 3\mu)} \\ &= (1 - M_1) \left(\frac{(\bar{W}_1(\mathcal{B}))^3(\rho_2 + \sigma_2)}{\left[4 \left(\frac{(\mu - \lambda)(2\mu + 1)}{\lambda^2 + 2} \right) (\bar{W}_1(\mathcal{B}))^2 - 2 \left(\frac{\bar{W}_2(\mathcal{B})}{2\bar{W}_1(\mathcal{B})} \right) (2\mu - \lambda + 2)^2 \right]} \right) \\ &\quad + \frac{\bar{W}_1(\mathcal{B})(\rho_2 - \sigma_2)}{4(6 - \lambda + 3\mu)} \\ &= \bar{W}_1(\mathcal{B}) \left(\left(\hbar(M_1) + \frac{1}{4(6 - \lambda + 3\mu)} \right) \rho_2 + \left(\hbar(M_1) - \frac{1}{4(6 - \lambda + 3\mu)} \right) \sigma_2 \right) \end{aligned}$$

where

$$\hbar(M_1) = \frac{(\bar{W}_1(\mathcal{B}))^3(1 - M_1)}{\left[4 \left(\frac{(\mu - \lambda)(2\mu + 1)}{\lambda^2 + 2} \right) (\bar{W}_1(\mathcal{B}))^2 - 2 \left(\frac{\bar{W}_2(\mathcal{B})}{2\bar{W}_1(\mathcal{B})} \right) (2\mu - \lambda + 2)^2 \right]}$$

Applying lemma 2.1, we get that

$$|a_3 - M_1 a_2^2| \leq \bar{W}_1(\mathcal{B}) \left((|\hbar(M)| + \frac{1}{4(6-\lambda+3\mu)}) |\rho_2| + (|\hbar(M)| - \frac{1}{4(6-\lambda+3\mu)}) |\sigma_2| \right)$$

In view of $|\rho_2| \leq 2$, $|\sigma_2| \leq 2$, from (1.4) and

if

$$0 \leq |\hbar(M)| \leq \frac{1}{4(6-\lambda+3\mu)}$$

We obtain

$$|a_3 - M_1 a_2^2| \leq \frac{(2\mathcal{B} - 1)}{2(6-\lambda+3\mu)}$$

and if

$$|\hbar(M)| \geq \frac{1}{4(6-\lambda+3\mu)}$$

We obtain

$$|a_3 - M_1 a_2^2| \leq 2(2\mathcal{B} - 1)|\hbar(M)|$$

The proof of theorem 3.1 is completed. ■

5. Related Corollaries

By refining the criteria used in our main findings from the preceding section, we derive other corollaries, such as:

Corollary 5.1. Assume $f \in \mathfrak{I}(1,1,\mathcal{B})$ that given by (1.1). Let M_1 be a real number, $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$. Then

$$|a_3 - M_1 a_2^2| \leq \begin{cases} \frac{(2\mathcal{B} - 1)}{16} & \text{if } 0 \leq |\hbar(M)| \leq \frac{1}{32} \\ 2(2\mathcal{B} - 1)|\hbar(M)| & \text{if } |\hbar(M)| \geq \frac{1}{32} \end{cases}$$

Corollary 5.2. Let $f \in \mathfrak{I}(2,1,\mathcal{B})$, f given by (1.1), M_1 be a real number, $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$. Then

$$|a_3 - M_1 a_2^2| \leq \begin{cases} \frac{(2\mathcal{B} - 1)}{24} & \text{if } 0 \leq |\hbar(M)| \leq \frac{1}{96} \\ 2(2\mathcal{B} - 1)|\hbar(M)| & \text{if } |\hbar(M)| \geq \frac{1}{96} \end{cases}$$

Corollary 5.3. Let f given by (1.11) be in the class $\mathfrak{I}(1,0,\mathcal{B})$. Permit M_1 be a real number, $\mathcal{B} \in (\frac{1}{2}, 1]$; $z, \omega \in U$ and $g = f^{-1}$. Then

$$|a_3 - M_1 a_2^2| \leq \begin{cases} \frac{(2\mathcal{B} - 1)}{18} & \text{if } 0 \leq |\hbar(M)| \leq \frac{1}{36} \\ 2(2\mathcal{B} - 1)|\hbar(M)| & \text{if } |\hbar(M)| \geq \frac{1}{36} \end{cases}$$

6. Conclusions

Numerous prominent mathematicians have recently focused on the study of polynomials and special functions due to their applicability across various mathematical and scientific fields. This work aims to use Euler polynomials to establish a notable subclass $\mathcal{H}(\mu, \lambda, \mathcal{B})$ of analytic and univalent functions. We have resolved the Fekete- Szegő problem and determined an upper limit estimate for the coefficients $|a_2|$ and $|a_3|$ of functions within this class of functions. The unresolved $|a_3 - M_2 a_2^2|$ problem and the significant upper limits $|a_2|$ and $|a_3|$ present compelling challenges that need to be addressed, together with the unresolved issue concerning $|a_k|$ where $k \geq 3$.

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The data are contained within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest

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